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Design of Experiments for Comparing Treatments With a Control: Tables of Optimal Allocations of Observations

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The problem considered is that of estimating simultaneously the differences between the means of $p \geq 2$ test treatments and the mean of a control treatment. For design purposes the population variances of all $p + 1$ treatments are regarded as known. Tables are given that provide the experimenter with a basis for determining the minimal total number of experimental units and the optimal allocation of these units among the $p + 1$ treatments, in order to make one-sided or two-sided joint confidence interval estimates of the differences between the mean of each of the test treatments and the mean of the control treatment. These intervals achieve a specified joint confidence coefficient $1 - \alpha$ for a specified allowance associated with the common width of the interval estimates. Comparisons with some competing allocation rules are also given.

KEY WORDS: Multiple comparisons with a control; Dunnett's procedure; Optimal allocation of observations.

1. INTRODUCTION

Consider the problem of comparing simultaneously $p \geq 2$ test treatments with a control treatment. The present article is concerned with certain design aspects of this problem—in particular, the design decision of how to "optimally" allocate experimental units among the test treatments and the control treatment. Dunnett (1955, 1964) considered the data analysis aspects of this problem and provided constants necessary to make joint $100(1 - \alpha)$ percent confidence statements (either one-sided or two-sided) concerning the differences between the mean of each of the test treatments and the mean of the control treatment when the common variance of the $p + 1$ treatments is unknown.

We begin by describing an example (see Dunnett 1955) that illustrates the underlying design considerations: It is known that the breaking strength of a fabric is affected by the chemical process with which it is treated. Suppose that researchers wish to compare the effects of three different chemical processes with the effect of a standard method of manufacture which is considered as the control treatment. In particular, suppose that their objective is to retain for further study only those chemical treatments which cause an improvement in the mean breaking strength over that of the control, and to discard the others. Thus they are interested in *one-sided* comparisons of means. The researchers wish to determine the minimum total number of observations necessary that would permit them to make the desired joint inferences with a specified control over the error probability for a given "allowance." Because the control treatment plays a pivotal role, in that every test treatment is compared to it, it is natural for the researchers to believe that more observations should be allocated to the control treatment than to each of the test treatments (assuming that the variability associated with each is the same). The question then is, "What is the optimal allocation of observations that would minimize the total number of observations to be taken?" The tables in the present article help to provide an answer to this question.

The researchers might proceed as follows in specifying statistical requirements: First, they recognize that they are dealing with a multiple comparisons problem. If the results are to be given in terms of confidence statements concerning the three differences between the means of the test treatments and the mean of the control treatment, then they may wish to achieve a *joint* confidence coefficient of $1 - \alpha = .95$ (say). Next, they specify the "allowance" (a term first introduced by Tukey 1953) on the error in the estimate of the difference between each treatment mean and the control mean. Here, allowance refers to the common "width" of the confidence intervals (see Section 2). For a specified joint confidence coefficient, a small (large) allowance would clearly require a large (small) total number of observations. Based on past experience with similar data, they may be willing to assume for *design* purposes that the variances of observations from the test treatments as well as from the control treatment are equal. If not enough past data are available to guide them, they may be able (e.g., based on the anticipated ranges of the observations) to specify an upper bound on the common value of the variance. This bound then can be used to design a *conservative* experiment. In that case, for *analysis* purposes the pooled sample estimate of the variance obtained from the data should be used. In Section 4 we shall provide a solution to the researchers' design problem using tables given in the present article; we shall also indicate there how they can carry out the design and analysis if they are able to specify only an upper bound on the common variance.

Strictly speaking, the tables herein are applicable only when all $p + 1$ variances have values (possibly unequal) that are *known* from past experience. The case of completely *unknown*, possibly unequal variances cannot be dealt with by the methods of the present article even when individual upper bounds on these variances can be postulated. This is so because in the analysis stage, use of estimates of these different variances leads to a Behrens–Fisher type of problem for which no exact solution is presently available (see, e.g., Tamhane 1977).

In this article we focus our attention on the problem of optimal allocation of experimental units among the test treatments and the control treatment to minimize the total size of the experiment subject to specified joint confidence coefficient and specified common allowance. (We recognize that additional considerations may also be present in planning any experiment, e.g., having enough data for a test of normality, possibly unequal costs of experimentation with different treatments, etc.; however, we do not deal with such problems here.)

As noted above, for design purposes the $p + 1$ variances are regarded as known and possibly unequal. Bechhofer (1969) (hereinafter referred to as B1) gave a solution to this problem for *one-sided* comparisons under the restriction that the variances of the sample means of the p test treatments are known and equal. Bechhofer and Nocturne (1972) (hereinafter referred to as B2) generalized these results to *two-sided* comparisons. Only small illustrative sets of tables of optimal allocations (all for $p = 2$) were given in B1 and B2. In the present article we give an extensive set of tables for $p = 2(1)10$ both for joint one-sided or joint two-sided comparisons based on the formulas given in B1 and B2. (See Remark 2.2 for the case $p = 1$.) For such comparisons these tables can be used to determine the smallest total number of observations necessary to guarantee selected joint confidence coefficients of (0.75, 0.90, 0.95, 0.99) for given specified common allowance; the tables also tell how to allocate these observations optimally among the $p + 1$ treatments.

Remark 1.1: The present paper (and each of the aforementioned papers) deals with the case in which a completely randomized design is to be employed. However, many practical situations may require the blocking of experimental units. If the block size is large enough to accommodate one replication of all of the test treatments and additional control treatments as well, then the optimal allocations in the present article can be used. If the blocks have a common size $k < p + 1$, that is, if the $p + 1$ treatments are to be compared in incomplete blocks of size k, then entirely new considerations are required to determine the optimal incomplete block design. This problem is considered in Bechhofer and Tamhane (1981); the optimal design tables for incomplete blocks of common size $k = 2$, 3, $p = k(1)6$ are given in Bechhofer and Tamhane (1982).

In Section 2 we introduce our notation and pose the optimal allocation problem both for one-sided and for two-sided comparisons. The tables containing constants necessary to implement the procedure are given in Sections 3 and 4 along with an explanation of how they are to be used. In Section 5 we make comparisons between the optimal allocations given herein and certain other allocation rules. Section 6 contains some concluding remarks. The formulas used in the computation of Tables 1 through 4, and details of the computations are given in the Appendix.

2. NOTATION AND STATEMENT OF THE OPTIMAL ALLOCATION PROBLEMS

Let the treatments be indexed by 0, 1, ..., p with 0 denoting the control treatment and 1, 2, ..., p denoting the $p \geq 2$ test treatments. We assume that the observations X_{ij} ($j = 1, 2, \ldots$) on the ith treatment are normally distributed with unknown mean μ_i and known variance σ_i^2 ($0 \leq i \leq p$), and that all observations are mutually independent. Based on $N_i \geq 1$ observations on the ith treatment ($0 \leq i \leq p$) it is desired to make either

1. A $100(1 - \alpha)$ percent joint *one-sided* confidence statement of the form

$$\{\mu_0 - \mu_i \leq \bar{x}_0 - \bar{x}_i + d \quad (1 \leq i \leq p)\}, \qquad (2.1)$$

or

2. A $100(1-\alpha)$ percent joint *two-sided* confidence statement of the form

$$\{\bar{x}_0 - \bar{x}_i - d \le \mu_0 - \mu_i \le \bar{x}_0 - \bar{x}_i + d \quad (1 \le i \le p)\}. \quad (2.2)$$

In (2.1) and (2.2), $\bar{x}_i$ is the observed value of the random variable $\bar{X}_i = \sum_{j=1}^{N_i} X_{ij}/N_i$ $(0 \le i \le p)$, and $d > 0$ is a specified common allowance.

The optimal allocation problem is that of finding the allocation vector $(N_0, N_1, \ldots, N_p)$, which for known $(\sigma_0^2, \sigma_1^2, \ldots, \sigma_p^2)$ and specified $1-\alpha$ and d, minimizes the total sample size $N = \sum_{i=0}^{p} N_i$ subject to

$$P\{\mu_0 - \mu_i \le \bar{X}_0 - \bar{X}_i + d \quad (1 \le i \le p)\} \ge 1 - \alpha \quad (2.3)$$

for *one-sided* comparisons, and

$$P\{\bar{X}_0 - \bar{X}_i - d \le \mu_0 - \mu_i \le \bar{X}_0 - \bar{X}_i + d \quad (1 \le i \le p)\} \ge 1 - \alpha \quad (2.4)$$

for *two-sided* comparisons. For both cases we denote the optimal allocation by $(\hat{N}_0, \hat{N}_1, \ldots, \hat{N}_p)$ and the smallest total sample size by $\hat{N} = \sum_{i=0}^{p} \hat{N}_i$ (the particular case under consideration being clear from the context).

Remark 2.1: This same optimal allocation $(\hat{N}_0, \hat{N}_1, \ldots, \hat{N}_p)$ maximizes the joint confidence coefficient for known $(\sigma_0^2, \sigma_1^2, \ldots, \sigma_p^2)$ and specified total sample size $N = \sum_{i=0}^{p} \hat{N}_i$ and d.

Continuous approximations to the probabilities (2.3) and (2.4) are obtained in B1 and B2, respectively, by letting

$$\gamma_i = N_i \Big/ \sum_{i=0}^{p} N_i \quad (0 \le i \le p), \quad (2.5)$$

and regarding the γ_i as nonnegative continuous variables satisfying $\sum_{i=0}^{p} \gamma_i = 1$. The solutions given in B1 and B2 provide optimal allocations for one-sided and two-sided comparisons under the restriction that the variances σ_i^2/N_i $(1 \le i \le p)$ of the *test* treatment means are equal; that is, $\sigma_i^2/N_i = \sigma_j^2/N_j$ $(i \ne j;\ 1 \le i, j \le p)$. The solution to the problem of optimal allocations for *one-sided* comparisons without any restriction on the σ_i^2/N_i $(1 \le i \le p)$ is given in Bechhofer and Turnbull (1971).

Under the restriction that the variances of the treatment means are equal, and using the continuous approximation, the probabilities (2.3) and (2.4) can be shown for given p to depend on $(\sigma_0^2, \sigma_1^2, \ldots, \sigma_p^2)$, N and d only through γ_0 (the proportion of the total number of observations taken on the control treatment) and two pure numbers

$$\lambda = d\sqrt{N}/\sigma_0 \quad (2.6)$$

and

$$\theta = \left(\sum_{i=1}^{p} \sigma_i^2\right) \Big/ \sigma_0^2. \quad (2.7)$$

(Note that in papers B1 and B2, β was used to denote the quantity θ defined in (2.7).) It sometimes might be reasonable to assume that the test variances σ_i^2 $(1 \le i \le p)$ are all equal to some multiple c (say) of the control variance σ_0^2; that is, $\theta = cp$. For this reason the θ values used in preparing the tables are chosen to be selected multiples of p. For given p and θ, and specified $1-\alpha$, the optimal solutions that we denote by $(\hat{\gamma}_0, \hat{\lambda})$ are uniquely determined. The simultaneous equations that yield these solutions are given in the Appendix.

The entries $(\hat{\gamma}_0, \hat{\lambda})$ in Tables 1 through 4 are to be used as follows: The σ_i^2 $(0 \le i \le p)$ are given as data of the problem; these determine θ, via (2.7). The experimenter specifies the allowance d, and the one-sided or two-sided joint confidence coefficient $1-\alpha$. Then p, θ, and the one-sided or two-sided $1-\alpha$ determine $(\hat{\gamma}_0, \hat{\lambda})$. From (2.6), the smallest total sample size $\hat{N}$ is then the smallest integer $\ge (\hat{\lambda}\sigma_0/d)^2$. The optimal allocations are given by $\hat{N}_0 = \hat{\gamma}_0 \hat{N}$ (to the nearest integer) and $\hat{N}_i = (\hat{N} - \hat{N}_0)\sigma_i^2/\theta\sigma_0^2$ (to the nearest integer) for $(1 \le i \le p)$; these *approximate integer* allocations that were obtained by using the continuous approximations will be very close to the *exact integer* allocations if $\hat{N}$ is large.

Remark 2.2: It should be noted that for $p = 1$ the optimal allocation both for one-sided and for two-sided comparisons is $\sigma_0/N_0 = \sigma_1/N_1$. Then $\hat{N} = \{(\sigma_0 + \sigma_1)z_\alpha/d\}^2$ and $\hat{N}_i = \hat{N}\sigma_i/(\sigma_0 + \sigma_1)$ $(i = 0, 1)$ for one-sided comparisons; the same expressions hold for two-sided comparisons with $z_{\alpha/2}$ replacing z_α. Here z_α is the upper α-point of the standard normal distribution.

3. DESCRIPTION OF THE TABLES

Tables 1 through 4 give values of $(\hat{\gamma}_0, \hat{\lambda})$, for $1-\alpha = 0.75, 0.90, 0.95$ and 0.99, respectively, both for one-sided and two-sided comparisons for $p = 2(1)10$. The tabulated values of $\hat{\gamma}_0$ are correct to within one in the third decimal place while the tabulated values of $\hat{\lambda}$ are rounded up in the third decimal place to guarantee a joint confidence coefficient $\ge 1-\alpha$ for the tabulated value of $\hat{\gamma}_0$. For each value of p and $1-\alpha$ the tabulations are given for $\theta = p/2$, p, $3p/2$, and $2p$. From (2.7) we see that the tables therefore can be used for the special case $\sigma_1^2 = \cdots = \sigma_p^2 = \sigma^2$ (say) when $\sigma^2 = c\sigma_0^2$ for $c = \frac{1}{2}, 1, \frac{3}{2}$, and 2. In particular, the $\theta = p$ column can be used for the special case $\sigma_0^2 = \sigma_1^2 = \cdots = \sigma_p^2$.

For fixed p, θ, and $1-\alpha$ an examination of the tables shows that $\hat{\gamma}_0$ and $\hat{\lambda}$ in the two-sided case are

Table 1. *Optimal Allocation*[1] *on the Control* ($\hat{\gamma}_0$) *and Associated* $\hat{\lambda}$ *to Achieve Joint Confidence Coefficient* $1 - \alpha = .75$

θ	Type of intervals	p = 2	3	4	5	6	7	8	9	10
p/2	1-sided	0.425 2.008	0.357 2.651	0.316 3.197	0.288 3.681	0.268 4.123	0.251 4.532	0.238 4.916	0.227 5.280	0.217 5.626
	2-sided	0.464 2.901	0.403 3.610	0.365 4.210	0.337 4.741	0.315 5.223	0.298 5.670	0.284 6.087	0.272 6.482	0.262 6.856
p	1-sided	0.352 2.482	0.292 3.351	0.257 4.094	0.233 4.757	0.215 5.363	0.202 5.927	0.190 6.456	0.181 6.958	0.173 7.436
	2-sided	0.387 3.541	0.332 4.495	0.298 5.309	0.274 6.033	0.255 6.693	0.240 7.305	0.228 7.879	0.218 8.422	0.209 8.938
3p/2	1-sided	0.312 2.842	0.257 3.882	0.225 4.775	0.204 5.574	0.188 6.306	0.175 6.987	0.165 7.628	0.157 8.235	0.150 8.814
	2-sided	0.344 4.027	0.293 5.168	0.262 6.144	0.240 7.014	0.223 7.810	0.209 8.549	0.198 9.242	0.189 9.898	0.181 10.523
2p	1-sided	0.284 3.145	0.233 4.328	0.204 5.346	0.184 6.258	0.170 7.096	0.158 7.877	0.149 8.611	0.142 9.307	0.135 9.971
	2-sided	0.314 4.435	0.267 5.732	0.238 6.844	0.217 7.838	0.201 8.747	0.189 9.592	0.179 10.386	0.171 11.138	0.163 11.854

1/ The upper entry in each cell in the body of the table is $\hat{\gamma}_0$ and the lower entry is $\hat{\lambda}$.

Table 2. *Optimal Allocation*[1] *on the Control* ($\hat{\gamma}_0$) *and Associated* $\hat{\lambda}$ *to Achieve Joint Confidence Coefficient* $1 - \alpha = .90$

θ	Type of intervals	p = 2	3	4	5	6	7	8	9	10
p/2	1-sided	0.468 3.149	0.409 3.876	0.371 4.491	0.343 5.035	0.322 5.529	0.304 5.986	0.290 6.414	0.278 6.817	0.267 7.201
	2-sided	0.480 3.830	0.424 4.606	0.387 5.261	0.360 5.840	0.338 6.366	0.321 6.851	0.307 7.305	0.294 7.733	0.284 8.139
p	1-sided	0.389 3.838	0.335 4.819	0.301 5.654	0.277 6.397	0.258 7.074	0.243 7.701	0.231 8.290	0.220 8.846	0.211 9.375
	2-sided	0.400 4.651	0.348 5.699	0.314 6.590	0.290 7.381	0.271 8.101	0.256 8.768	0.244 9.393	0.233 9.983	0.224 10.544
3p/2	1-sided	0.345 4.364	0.295 5.537	0.263 6.540	0.241 7.434	0.224 8.251	0.211 9.008	0.200 9.720	0.190 10.392	0.182 11.033
	2-sided	0.354 5.278	0.306 6.532	0.275 7.603	0.253 8.556	0.236 9.425	0.222 10.231	0.211 10.986	0.201 11.700	0.193 12.379
2p	1-sided	0.315 4.805	0.268 6.140	0.238 7.284	0.218 8.305	0.202 9.240	0.190 10.107	0.179 10.922	0.171 11.693	0.163 12.426
	2-sided	0.323 5.804	0.278 7.233	0.249 8.456	0.228 9.544	0.212 10.538	0.200 11.461	0.189 12.326	0.180 13.145	0.173 13.924

1/ The upper entry in each cell in the body of the table is $\hat{\gamma}_0$ and the lower entry is $\hat{\lambda}$.

Table 3. *Optimal Allocation*[1] *on the Control* ($\hat{\gamma}_0$) *and Associated* $\hat{\lambda}$ *to Achieve Joint Confidence Coefficient* $1 - \alpha = .95$

θ	Type of intervals	p = 2	3	4	5	6	7	8	9	10
p/2	1-sided	0.480 3.830	0.424 4.606	0.387 5.261	0.359 5.841	0.338 6.366	0.321 6.852	0.306 7.306	0.294 7.734	0.283 8.140
	2-sided	0.487 4.423	0.432 5.242	0.396 5.933	0.369 6.543	0.348 7.097	0.330 7.608	0.316 8.086	0.303 8.536	0.292 8.963
p	1-sided	0.399 4.651	0.348 5.700	0.314 6.591	0.290 7.382	0.271 8.103	0.256 8.770	0.243 9.395	0.233 9.986	0.223 10.547
	2-sided	0.405 5.360	0.354 6.469	0.321 7.411	0.297 8.247	0.278 9.007	0.263 9.711	0.250 10.369	0.239 10.990	0.230 11.581
3p/2	1-sided	0.354 5.278	0.305 6.534	0.274 7.605	0.252 8.559	0.235 9.429	0.221 10.235	0.210 10.991	0.200 11.705	0.192 12.385
	2-sided	0.358 6.077	0.311 7.407	0.281 8.540	0.258 9.548	0.241 10.466	0.227 11.317	0.216 12.114	0.206 12.866	0.198 13.582
2p	1-sided	0.323 5.806	0.277 7.235	0.248 8.458	0.227 9.548	0.211 10.543	0.199 11.467	0.188 12.333	0.180 13.152	0.172 13.931
	2-sided	0.327 6.680	0.282 8.195	0.254 9.490	0.233 10.642	0.217 11.694	0.204 12.669	0.193 13.582	0.185 14.445	0.177 15.267

1/ The upper entry in each cell in the body of the table is $\hat{\gamma}_0$ and the lower entry is $\hat{\lambda}$.

Table 4. *Optimal Allocation*[1] *on the Control* ($\hat{\gamma}_0$) *and Associated* $\hat{\lambda}$ *to Achieve Joint Confidence Coefficient* $1 - \alpha = .99$

θ	Type of intervals	p = 2	3	4	5	6	7	8	9	10
p/2	1-sided	0.492 5.115	0.439 5.986	0.403 6.720	0.376 7.369	0.355 7.956	0.338 8.499	0.323 9.005	0.310 9.482	0.299 9.934
	2-sided	0.494 5.589	0.442 6.496	0.406 7.261	0.380 7.936	0.358 8.548	0.341 9.112	0.326 9.639	0.314 10.136	0.303 10.606
p	1-sided	0.409 6.190	0.359 7.373	0.326 8.377	0.302 9.267	0.283 10.076	0.268 10.824	0.255 11.524	0.244 12.184	0.235 12.811
	2-sided	0.410 6.758	0.361 7.993	0.329 9.042	0.304 9.970	0.286 10.814	0.270 11.595	0.257 12.324	0.246 13.012	0.237 13.665
3p/2	1-sided	0.362 7.012	0.315 8.433	0.285 9.644	0.262 10.719	0.245 11.697	0.231 12.603	0.220 13.451	0.210 14.252	0.201 15.013
	2-sided	0.363 7.652	0.317 9.140	0.287 10.405	0.264 11.528	0.247 12.549	0.233 13.495	0.221 14.380	0.211 15.215	0.203 16.008
2p	1-sided	0.330 7.704	0.286 9.327	0.257 10.710	0.236 11.941	0.220 13.063	0.207 14.102	0.197 15.075	0.187 15.994	0.180 16.867
	2-sided	0.331 8.406	0.287 10.105	0.259 11.553	0.238 12.839	0.222 14.011	0.209 15.096	0.198 16.112	0.189 17.070	0.181 17.981

1/ The upper entry in each cell in the body of the table is $\hat{\gamma}_0$ and the lower entry is $\hat{\lambda}$.

always greater than the corresponding $\hat{\gamma}_0$ and $\hat{\lambda}$ in the one-sided case. For fixed p and θ, in both cases $\hat{\gamma}_0$ increases with $1-\alpha$ and approaches the limit $1/(1+\sqrt{\theta})$ as $1-\alpha$ approaches unity (and hence $\hat{\gamma}_0/\hat{\gamma}_i \to \sqrt{\theta}\,\sigma_0^2/\sigma_i^2$ for $1 \le i \le p$). This limiting result has been proven analytically in B1 and B2 for the one-sided and two-sided cases, respectively. For $\sigma_0^2 = \sigma_1^2 = \cdots = \sigma_p^2$ this gives the limiting result that $\hat{\gamma}_0/\hat{\gamma}_i \to \sqrt{p}$ $(1 \le i \le p)$. Note that Dunnett (1955, p. 1107) recommended that $N_0/N_i = \sqrt{p}$ $(1 \le i \le p)$; this recommendation had been made earlier by Finney (1952) and other authors (but not in the context of *multiple comparisons* with a control).

4. USE OF THE TABLES

To illustrate the use of the tables we return to the example described in Section 1. For that example, assume $\sigma_i = 5$ $(0 \le i \le 3)$ and $d = 5$; then $\theta = p = 3$. For one-sided intervals with $d = 5$ and $1-\alpha = .95$ we find from Table 3 that $\hat{\gamma}_0 = .348$ and $\hat{\lambda} = 5.700$. Hence $\hat{N} = \langle\{(5.700)5/5\}^2\rangle = \langle 32.49\rangle = 33$ (where $\langle x\rangle$ denotes the smallest integer $\ge x$) and $\hat{N}_0 = 12$, $\hat{N}_1 = \hat{N}_2 = N_3 = 7$. Thus by taking 7 observations on each of the test treatments and 12 observations on the control treatment the researchers can obtain the desired joint confidence interval estimates using the smallest possible total sample size.

If $\hat{N} = 33$ seems too high in relation to the resources available to the researchers, then they must make trade-offs by either settling for larger allowance and/or smaller joint confidence coefficient.

For values of θ and $1-\alpha$ not given in the present article, quadratic interpolation should give results sufficiently accurate for practical purposes. It might be noted that $\hat{\gamma}_0$ is relatively insensitive to $1-\alpha$ but $\hat{\lambda}$ is naturally quite sensitive to it. Also, the total sample size $\hat{N}$ is quite sensitive to the assumed σ_i^2 and d. The assumed σ_i^2 $(0 \le i \le p)$ affect $\hat{N}$ in two ways: (a) directly through σ_0^2 via (2.6), and (b) through $\theta = \sum_{i=1}^p \sigma_i^2/\sigma_0^2$. A small assumed value of σ_0^2 leads to larger θ (for fixed $\sigma_1^2, \ldots, \sigma_p^2$) and hence larger $\hat{\lambda}$, but since $\hat{N}$ is proportional to σ_0^2, the final value of $\hat{N}$ may turn out to be smaller.

If the experimenter is prepared to assume that $\sigma_i^2 = \sigma^2$ $(0 \le i \le p)$ where the actual value of σ^2 is *unknown*, and believes that $\sigma^2 \le \sigma_U^2$ where σ_U^2 is known, then this information can be used in *designing* the experiment, for example, acting as if $\sigma^2 = \sigma_U^2$ leads to a conservative choice of $\hat{N}$. However, *after* the experiment has been conducted, when the results are being summarized, the common unknown σ^2 should be estimated using the pooled data, the usual unbiased estimate s^2 being based on $\nu = \hat{N} - (p+1)$ df if a completely randomized design is used. The estimate then should be used with Dunnett's (1955) formulas for joint confidence statements (analogous to statements 1 and 2 of (2.1) and (2.2), respectively):

3. A $100(1-\alpha)$ percent joint *one-sided* confidence statement

$$\{\mu_0 - \mu_1 \le \bar{x}_0 - \bar{x}_i - + t_{\nu,p,\rho}^{(\alpha)} s\sqrt{(1/\hat{N}_0) + (1/\hat{N}_1)} \quad (1 \le i \le p)\} \tag{4.1}$$

or

4. A $100(1-\alpha)$ percent joint *two-sided* confidence statement

$$\{\bar{x}_0 - \bar{x}_i - t_{\nu,p,\rho}^{\prime(\alpha)} s\sqrt{(1/\hat{N}_0) + (1/\hat{N}_1)} \le \mu_0 - \mu_i \le \bar{x}_0 - \bar{x}_i + t_{\nu,p,\rho}^{\prime(\alpha)} s\sqrt{(1/\hat{N}_0) + (1/\hat{N}_1)} \quad (1 \le i \le p)\}. \tag{4.2}$$

Here $t_{\nu,p,\rho}^{(\alpha)}$ $(t_{\nu,p,\rho}^{\prime(\alpha)})$ is the upper α equicoordinate point of the p-variate t-distribution (p-variate $|t|$-distribution) with df ν and equal correlations $\rho = \hat{N}_1/(\hat{N}_0 + \hat{N}_1)$; tables of $t_{\nu,p,\rho}^{(\alpha)}$ are given for selected ρ by Krishnaiah and Armitage (1966), while tables of $t_{\nu,p,\rho}^{\prime(\alpha)}$ are given for selected ρ by Hahn and Hendrickson (1971).

5. COMPARISONS WITH OTHER ALLOCATION RULES

In this section we compare the optimal allocation rule, which we denote by $\hat{R}$, with three other rules: (a) equal allocation rule R_{EQ} (explained below), (b) Dunnett's (1955) allocation rule R_{D} mentioned at the end of Section 3, and (c) Bechhofer and Turnbull's (1971) unrestricted optimal allocation rule R_{BT}, mentioned in the paragraph following Remark 2.1. The comparisons will be based on the sample sizes $\hat{N}$ and N required by $\hat{R}$ and the competing rule R, respectively, to guarantee the same joint one-sided (two-sided) confidence coefficient $1-\alpha$ using (2.3) (using (2.4)) for given $(\sigma_0^2, \sigma_1^2, \ldots, \sigma_p^2)$ and specified d. The value of $\hat{N}$ required in these comparisons is given by

$$\hat{N} = \langle(\hat{\lambda}\sigma_0/d)^2\rangle, \tag{5.1}$$

and $\hat{\lambda}$ is chosen from the appropriate table. Formulas for the N-values required by the three competing rules are given below. In each case we use these to make numerical comparisons with $\hat{N}$.

(a) *Equal allocation rule* R_{EQ}: For this rule the N_i $(0 \le i \le p)$ are chosen equal to $N_{\mathrm{EQ}}/(p+1)$, where N_{EQ} is the total sample size required by R_{EQ} to guarantee the specified requirement on the joint confidence statement. We give the formula for N_{EQ} in the important special case $\sigma_0^2 = \sigma_1^2 = \cdots = \sigma_p^2 = \sigma^2$ (say).

For *one-sided* comparison

$$N_{\mathrm{EQ}} = (p+1)\langle 2\{t_{\infty,p,1/2}^{(\alpha)}(\sigma/d)\}^2\rangle \tag{5.2}$$

where $t_{\infty,p,1/2}^{(\alpha)}$ is the upper α-point of the distribution of the maximum of p equicorrelated standard normal random variables with common correlation $\rho = \frac{1}{2}$. The values of $t^{(\alpha)}_{\infty, p, 1/2}$ have been tabulated for selected p and $1 - \alpha$ by Gupta, Nagel, and Panchapakesan (1973).

For *two-sided* comparisons $t^{(\alpha)}_{\infty, p, 1/2}$ in (5.2) is replaced by $t'^{(\alpha)}_{\infty, p, 1/2}$, the upper α-point of the distribution of the maximum of the *absolute* values of p equicorrelated standard normal random variables with common correlation $\rho = \frac{1}{2}$; the values of $t'^{(\alpha)}_{\infty, p, 1/2}$ have been tabulated for selected p and $1 - \alpha$ by Odeh (1982).

Some representative values of $\hat{N}$ and N_{EQ} are given in Table 5 for $\sigma/d = 5$. It can be seen that the relative savings $(N_{\mathrm{EQ}} - \hat{N})/\hat{N}$ as well as the absolute savings $(N_{\mathrm{EQ}} - \hat{N})$ increase with p and $1 - \alpha$; for two-sided comparisons both relative and absolute savings are greater in each case than those for one-sided comparisons. It should be noted from (5.1) and (5.2) that the relative saving (ignoring the integer restrictions on $\hat{N}$ and N_{EQ}) is independent of σ/d while the absolute saving is directly proportional to $(\sigma/d)^2$.

(b) *Dunnett's allocation rule* R_{D}: Again for convenience, we consider the important special case $\sigma_0^2 = \sigma_1^2 = \cdots = \sigma_p^2 = \sigma^2$ (say). R_{D} chooses the N_i $(1 \le i \le p)$ equal to $N_0/\sqrt{p} = N_{\mathrm{D}}/\sqrt{p}(1 + \sqrt{p})$, where N_{D} is the total sample size required by R_{D} to guarantee the specified requirement on the joint confidence statement. For *one-sided* comparisons

$$N_{\mathrm{D}} = \langle \{(1 + \sqrt{p})(\sigma/d) t^{(\alpha)}_{\infty, p, \rho*}\}^2 \rangle, \qquad (5.3)$$

where $\rho* = (1 + \sqrt{p})^{-1}$. For *two-sided* comparisons $t^{(\alpha)}_{\infty, p, \rho*}$ is replaced by $t'^{(\alpha)}_{\infty, p, \rho*}$.

Tables of $t^{(\alpha)}_{\infty, p, \rho*}$ and $t'^{(\alpha)}_{\infty, p, \rho*}$ are available only for $p = 4$ when $\rho* = \frac{1}{3}$, and for $p = 9$ when $\rho* = \frac{1}{4}$; see Gupta, Nagel, and Panchapakesan (1973) for the tables of $t^{(\alpha)}_{\infty, p, \rho*}$ and Odeh (1982) for the tables of $t'^{(\alpha)}_{\infty, p, \rho*}$. For other values of p, interpolation in ρ becomes necessary. In Table 6 we have given representative values of $\hat{N}$ and N_{D} only for $p = 4$ and $p = 9$. It can be seen that R_{D} gives N-values quite close to the optimum $\hat{N}$ given by $\hat{R}$ for large values of $1 - \alpha$ $(\ge .95)$. This is not surprising in view of the fact that as $1 - \alpha$ approaches unity, $\hat{R}$ approaches R_{D}. However, for moderate values of $1 - \alpha$ ($\sim$ .75 to .90) the absolute saving $(N_{\mathrm{D}} - \hat{N})$, which is proportional to $(\sigma/d)^2$, can be large if σ/d is large.

Table 5. *Values of* $\hat{N}$ *and* N_{EQ} $(\sigma_0^2 = \ldots = \sigma_p^2 = \sigma^2;$ $\theta = p;\ \sigma/d = 5)$[1]

$1 - \alpha$	Type of Comparison	p = 2	p = 5	p = 10
0.75	1 - sided	154 156	566 582	1383 1474
	2 - sided	314 318	910 978	1998 2277
0.95	1 - sided	541 552	1363 1500	2781 3366
	2 - sided	719 735	1701 1896	3353 4059
0.99	1 - sided	958 984	2147 2418	4103 5060
	2 - sided	1142 1173	2485 2814	4668 5797

[1] The upper entry in each cell is $\hat{N}$ and the lower entry is N_{EQ}.

Table 6. *Values of* $\hat{N}$ and N_{D} $(\sigma_0^2 = \ldots = \sigma_p^2 = \sigma^2;$ $\theta = p;\ \sigma/d = 5)$[1]

$1 - \alpha$	Type of Comparison	p = 4	p = 9
0.75	1-sided	419 429	1211 1238
	2-sided	705 709	1773 1782
0.95	1-sided	1086 1088	2493 2497
	2-sided	1373 1374	3020 3022
0.99	1-sided	1755 1755	3711 3712
	2-sided	2044 2044	4233 4233

[1] The upper entry in each cell is $\hat{N}$ and the lower entry is N_{D}.

(c) *Bechhofer and Turnbull's unrestricted optimal allocation rule* R_{BT}: The optimal allocation rule $\hat{R}$ of the present article computes the allocations under the restriction that the variances of the treatment means σ_i^2/N_i $(1 \le i \le p)$ are equal. For most practical applications this is a reasonable restriction. However, it is of some theoretical interest to determine how much is lost in terms of the increased total sample size because of this restriction. We emphasize that if the σ_i^2 $(1 \le i \le p)$ are equal, then the unrestricted and restricted optimal allocations are *identical* and nothing is lost because of the imposition of the restriction.

For *one-sided* comparisons the equations for finding the unrestricted optimal allocation and the associated minimal total sample size N_{BT} are given in Bechhofer and Turnbull (1971) for specified $1 - \alpha$ and d and given $(\sigma_0^2, \sigma_1^2, \ldots, \sigma_p^2)$. Using these equations we have made a sample calculation of N_{BT} for $p = 2$, $\sigma_0^2 = 1$, $\sigma_1^2 = \frac{1}{10}$, $\sigma_2^2 = \frac{9}{10}$ (i.e., $\theta = 1 = p/2$), $\sigma_0/d = 5$ and $1 - \alpha = .75$, .95, and .99; the corresponding $\hat{N}$-values were calculated from (5.1). The results are shown in Table 7.

Table 7. *Values of $\hat{N}$ and N_{BT} for One-Sided Comparisons* ($p = 2$; $\sigma_0^2 = 1$, $\sigma_1^2 = 1/10$, $\sigma_2^2 = 9/10$; $\sigma_0/d = 5$)

$1 - \alpha$	$\hat{N}$	N_{BT}
0.75	101	88
0.95	367	338
0.99	655	614

It can be seen that substantial savings in total sample size are possible using the unrestricted optimal allocation if the σ_i^2 $(1 \le i \le p)$ are *highly unequal*, which they are in the present example. However, it is not feasible to give tables of unrestricted optimal allocations not only because they are much harder to compute and require tabulation of $p + 1$ quantities—λ and $(\gamma_0, \gamma_1, \ldots, \gamma_{p-1})$—but also because a separate calculation must be made for every $(\sigma_1^2/\sigma_0^2, \ldots, \sigma_p^2/\sigma_0^2)$-vector.

6. CLOSING REMARKS

The tables in this article should be useful in the design of experiments for comparing several test treatments with a control. The tables enable the experimenter to determine the minimum total sample size necessary in order to make specified one-sided or two-sided joint confidence interval estimates of the differences between the means of each of the test treatments and the mean of the control treatment; the tables also tell the experimenter how to allocate the observations optimally among the test treatments and the control treatment (under the restriction that the variances of the test treatment means are equal). Comparisons with certain other allocation rules indicate that substantial savings are possible using the optimal allocations given herein.

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APPENDIX: FORMULAS FOR OPTIMAL ALLOCATION, AND DETAILS OF COMPUTATION

A.1 Formulas for Optimal Allocation for One-Sided Comparisons (Reference B1)

Let $\Phi(\cdot)$ and $\phi(\cdot)$ denote the standard normal distribution and density function, respectively, and let $\Phi_k(\cdot \mid \rho)$ denote the equicoordinate k-variate standard normal distribution function with common correlation ρ. Then the $(\hat{\gamma}_0, \hat{\lambda})$ given in Tables 1 through 4 are the unique solutions of the following simultaneous equations (A.1) and (A.2):

$$\int_{-\infty}^{\infty} \Phi^p\left[\left(\frac{x}{\sqrt{\gamma}} + \lambda\right)\left(\frac{1-\gamma}{\theta}\right)^{1/2}\right] d\Phi(x) = 1 - \alpha, \quad \text{(A.1)}$$

$$[(1-\theta)\gamma^2 - 2\gamma + 1]\tau\Phi_{p-1}[\tau \mid (1-\gamma)/\{2(1-\gamma)+\gamma\theta\}] - \frac{(p-1)\gamma(1-\gamma)\theta}{2(1-\gamma)+\gamma\theta}\phi(\tau) \times \Phi_{p-2}\left[\tau\left\{\frac{1-\gamma+\gamma\theta}{3(1-\gamma)+\gamma\theta}\right\}^{1/2} \middle| \frac{1-\gamma}{3(1-\gamma)+\gamma\theta}\right] = 0, \quad \text{(A.2)}$$

where

$$\tau = \lambda\gamma\left\{\frac{\theta(1-\gamma)}{[1-\gamma+\gamma\theta][2(1-\gamma)+\gamma\theta]}\right\}^{1/2}.$$

A.2 Formulas for Optimal Allocation for Two-Sided Comparisons (Reference B2)

Here the $(\hat{\gamma}_0, \hat{\lambda})$ given in Tables 1 through 4 are the unique solutions of the following simultaneous equations (A.3) and (A.4):

$$\int_{-\infty}^{\infty} \left\{\Phi\left[\left(\frac{x}{\sqrt{\theta}} + \lambda\right)\left(\frac{1-\gamma}{\theta}\right)^{1/2}\right] - \Phi\left[\left(\frac{x}{\sqrt{\gamma}} - \lambda\right)\left(\frac{1-\gamma}{\theta}\right)^{1/2}\right]\right\}^p d\Phi(x) = 1 - \alpha, \quad \text{(A.3)}$$

$$\frac{\lambda[(1-\theta)\gamma^2 - 2\gamma + 1]D_1}{(1-\gamma+\gamma\theta)^{1/2}} - \frac{(p-1)[\theta(1-\gamma)]^{1/2}}{[2(1-\gamma)+\gamma\theta]^{1/2}} \times \left\{\phi(\tau)D_2 - \phi\left[\tau\left\{\frac{2(1-\gamma)+\gamma\theta}{\gamma\theta}\right\}\right]D_3\right\} = 0, \quad \text{(A.4)}$$

where

$$D_1 = \Phi_{p-1}(-\Delta_1\tau_1, \tau_1 \mid (1-\gamma)/\{2(1-\gamma)+\gamma\theta\}),$$

$$D_2 = \Phi_{p-2}(-\Delta_2\tau_2, \tau_2 \mid (1-\gamma)/\{3(1-\gamma)+\gamma\theta\}),$$

$$D_3 = \Phi_{p-2}(-\tau_3, \tau_3 \mid (1-\gamma)/\{3(1-\gamma)+\gamma\theta\}),$$

$$\tau_1 = \tau,$$

$$\tau_2 = \tau[(1-\gamma+\gamma\theta)/\{3(1-\gamma)+\gamma\theta\}]^{1/2},$$

$$\tau_3 = \Delta_1\tau_2,$$

$$\Delta_1 = \{2(1-\gamma)+\gamma\theta\}/\gamma\theta,$$

$$\Delta_2 = \{4(1-\gamma)+\gamma\theta\}/\gamma\theta,$$

and

$$\Phi_k(a, b \mid \rho) = P\{a \le Z_i \le b \quad (1 \le i \le k)\},$$

where the Z_i are standard normal with $\text{corr}\{Z_i, Z_j\} = \rho$ for $i \neq j$, $1 \leq i, j \leq k$.

A.3 Details of Computation

The IMSL (1978) subroutine ZSYSTM was used to solve the pairs of simultaneous equations (A.1), (A.2) and (A.3), (A.4). The stopping criteria used in arriving at the final solutions were the following: (a) the difference between the left and the right sides of each equation is less than 1×10^{-6} or (b) in two successive iterations the corresponding trial values of $\hat{\gamma}_0$ and $\hat{\lambda}$ do not differ in the first six significant digits.

To evaluate a quantity of the form $\Phi_k(a, b \mid \rho)$ (which includes $\Phi_k(b \mid \rho)$ as a special case for $a = -\infty$) the following iterated integral representation (see equation (2) of Bechhofer and Tamhane 1974) was used:

$$\Phi_k(a, b \mid \rho) = \int_{-\infty}^{\infty} \left\{ \Phi\left[\frac{x\rho^{1/2} + b}{(1-\rho)^{1/2}}\right] - \Phi\left[\frac{x\rho^{1/2} + a}{(1-\rho)^{1/2}}\right] \right\}^k d\Phi(x).$$

For $p = 2$ the quantity $\Phi_{p-1}(a, b \mid \rho)$ reduces to $\Phi(b) - \Phi(a)$ and $\Phi_{p-2}(a, b \mid \rho) = \Phi_0(a, b \mid \rho) = 1$. Thus the evaluation of the various expressions is particularly simple for $p = 2$.

To evaluate $\Phi(\cdot)$ the formula (26.2.17) given in Abramowitz and Stegun (1964) was used; this formula is accurate to within $\pm 7.5 \times 10^{-8}$. The Romberg quadrature method (Davis and Rabinowitz 1967, p. 166) was used to evaluate the various integrals. All of the calculations were done on a CDC 6600 computer at Northwestern University.

The tabulated values of $\hat{\gamma}_0$ are rounded off in the third decimal place while the values of $\hat{\lambda}$ are rounded up in the third decimal place (to insure a joint confidence coefficient $\geq 1 - \alpha$).

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REFERENCES

ABRAMOWITZ, M. and STEGUN, I. A. (1964), *Handbook of Mathematical Functions*, National Bureau of Standards Applied Mathematics Series 55, Washington, D.C.: U.S. Government Printing Office.

BECHHOFER, R. E. (1969), "Optimal Allocation of Observations When Comparing Several Treatments with a Control," in *Multivariate Analysis—II*, ed. P. R. Krishnaiah, New York: Academic Press, 463–473.

BECHHOFER, R. E., and NOCTURNE, D. J. (1972), "Optimal Allocation of Observations When Comparing Several Treatments With a Control, II: 2-Sided Comparisons," *Technometrics*, 14, 423–436.

BECHHOFER, R. E., and TAMHANE, A. C. (1974), "An Iterated Integral Representation for a Multivariate Normal Integral Having Block Covariance Structure," *Biometrika*, 61, 615–619.

——— (1981), "Incomplete Block Designs for Comparing Treatments With a Control: General Theory," *Technometrics*, 23, 45–57.

——— (1982), *Tables of Admissible and Optimal Balanced Treatment Incomplete Block (BTIB) Designs for Comparing Treatments With a Control*, to appear in *Selected Tables in Mathematical Statistics*.

BECHHOFER, R. E., and TURNBULL, B. W. (1971), "Optimal Allocation of Observations When Comparing Several Treatments with a Control, III: Globally Best One-Sided Intervals for Unequal Variances," in *Statistical Decision Theory and Related Topics*, eds. S. S. Gupta and J. Yackel, New York: Academic Press, 41–78.

DAVIS, P. J., and RABINOWITZ, P. (1967), *Numerical Integration*, Waltham, Mass.: Blaisdell Publishing Company.

DUNNETT, C. W. (1955), "A Multiple Comparison Procedure for Comparing Several Treatments With a Control," *Journal of the American Statistical Association*, 50, 1096–1121.

——— (1964), "New Tables for Multiple Comparisons With a Control," *Biometrics*, 20, 482–491.

FINNEY, D. J. (1952), *Statistical Methods in Biological Assay*, New York: Hafner.

GUPTA, S. S., NAGEL, K., and PANCHAPAKESAN, S. (1973), "On the Order Statistics From Equally Correlated Normal Random Variables," *Biometrika*, 60, 403–413.

HAHN, G. J., and HENDRICKSON, R. W. (1971), "A Table of Percentage Points of the Distribution of the Largest Absolute Value of k Student t Variates and Its Application," *Biometrika*, 58, 323–332.

INTERNATIONAL MATHEMATICAL AND STATISTICAL LIBRARIES (1978), *IMSL Manual* (vols. I and II), Houston, Texas: IMSL.

KRISHNAIAH, P. R., and ARMITAGE, J. V. (1966), "Tables for Multivariate t-Distribution," *Sankhyā*, Ser. B, 28, 31–56.

ODEH, R. E. (1982), "Tables of Percentage Points of the Distribution of the Maximum Absolute Value of Equally Correlated Normal Random Variables," *Communications in Statistics*, B11, 65–88.

TAMHANE, A. C. (1977), "Multiple Comparisons in Model I One-Way ANOVA With Unequal Variances," *Communications in Statistics*, A6, 15–32.

TUKEY, J. W. (1953), "The Problem of Multiple Comparisons," Mimeographed Notes, Princeton University.